

Remarks on minimal rational curves on moduli spaces of stable bundles

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Abstract

Let M be the moduli space of stable bundles of rank 2 and with fixed determinant \mathcal{L} of degree d on a smooth projective curve C of genus $g \geq 2$. When $g = 3$ and d is even, we prove, for any point $[W] \in M$, there is a minimal rational curve passing through $[W]$, which is not a Hecke curve. This complements a theorem of Xiaotao Sun.

1 Introduction

Let C be a smooth projective curve of genus $g \geq 2$ and \mathcal{L} a line bundle on C of degree d . Let $M := SU_C(r, \mathcal{L})$ be the moduli space of stable vector bundles of rank r and with the fixed determinant \mathcal{L} , which is a smooth quasi-projective Fano-variety with $Pic(M) = \mathbb{Z} \cdot \Theta$ and $-K_M = 2(r, d)\Theta$, where Θ is an ample divisor ([8] [1]). For any rational curve $\phi : \mathbb{P}^1 \rightarrow M$, we can define its degree $\deg \phi^*(-K_M)$ with respect to the ample anti-canonical line bundle $-K_M$.

In [9], Xiaotao Sun determines all the rational curves of minimal degree passing through a generic point of M except in the case of $g = 3$, $r = 2$ and d is even.

Theorem 1.1. *(Theorem 1 of [9]) If $g \geq 3$, then any rational curve $\phi : \mathbb{P}^1 \rightarrow M$ passing through the generic point has degree at least $2r$. It has degree $2r$ if and only if it is a Hecke curve **unless** $g = 3$, $r = 2$, **and** d is even.*

It implies that all the rational curves of $(-K_M)$ -degree smaller than $2r$, called *small rational curves*, must lie in a proper closed subset ([3], [4]). In this note, we remark that the condition in Sun's Theorem is necessary:

Theorem 1.2. *If $g = 3$, $r = 2$ and d is even, then, for any point $[W] \in M$, there exists a rational curve of degree 4 passing through it, which is not a Hecke curve.*

Recall that, by Lemma 2.1 of [9], any rational curve $\phi : \mathbb{P}^1 \rightarrow M$ is defined by a vector bundle E on $f : X = C \times \mathbb{P}^1 \rightarrow C$. **If E is semi-stable on generic fiber** $X_\xi = f^{-1}(\xi)$, according to the arguments of [9], there is a finite set $S \subset C$ of points and a vector bundle V on C such that E is suited in the exact sequence

$$0 \rightarrow f^*V \rightarrow E \rightarrow \bigoplus_{p \in S} \mathcal{Q}_p \rightarrow 0$$

where \mathcal{Q}_p is a vector bundle on $X_p = \{p\} \times \mathbb{P}^1$. The curves defined by such E was called of **Hecke type** in [10] (since a Hecke curve by definition is defined by a vector

⁰ Supported by the National Natural Science Foundation of China (Grant No. 11401330)

bundle E suited in $0 \rightarrow f^*V \rightarrow E \rightarrow \mathcal{O}_{X_p}(-1) \rightarrow 0$). If E is not semi-stable on generic fiber X_ξ (curves defined by such E was called of **split type** in [10]) and the curve has minimal degree (i.e. a line), then E must suite in

$$0 \rightarrow f^*V_1 \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(1) \rightarrow E \rightarrow f^*V_2 \rightarrow 0$$

where $\pi : X \rightarrow \mathbb{P}^1$ is the projection and V_1, V_2 are stable vector bundles on C of rank r_1, r_2 and degree d_1, d_2 satisfying $r_1d - rd_1 = (r, d)$.

The rational curves we constructed in Theorem 1.2 are of split type (thus they are not Hecke curves). We in fact have a more general result. Let $M = \mathcal{SU}_C(2, \mathcal{L})$ be the moduli space of rank two stable bundles with fixed determinant \mathcal{L} on a smooth projective curve C of genus $g \geq 3$. Let $M_s \subset M$ be the locus of stable bundles $[W] \in M$ with Segre invariant $s(W) = s$. Then we have

Theorem 1.3. *When d is even, for any $[W] \in M_2$, there is a rational curve of split type passing through it, which has degree 4. If d is odd, for any $[W] \in M_1$, there is a rational curve of split type passing through it, which has degree 2.*

When $g = 3$ and d is even, we have $M_2 = M$ (see Lemma 3.1). Thus Theorem 1.2 is a corollary of Theorem 1.3.

2 Rational curves of split type

Let C be a smooth irreducible projective curve with genus $g \geq 2$ over an algebraically closed field, W be a stable bundle of rank r and with determinant \mathcal{L} over C . If there is a stable subbundle V_1 of W such that

$$r_1d - d_1r = (r, d), \tag{1}$$

where $r_1 = \text{rank}V_1$, $d_1 = \text{deg}V_1$ and $d = \text{deg}W$. Let $V_2 := W/V_1$ be the quotient bundle, then W fits a non-trivial extension

$$0 \rightarrow V_1 \rightarrow W \rightarrow V_2 \rightarrow 0. \tag{2}$$

It is known that there is a family of vector bundles $\{\mathcal{E}_p\}_{p \in P}$ on C parametrized by $P = \mathbb{P}\text{Ext}^1(V_2, V_1)$ so that for each $p \in P$, the \mathcal{E}_p is isomorphic to the bundle obtained as the extension of V_2 by V_1 given by p (see Lemma 2.3 of [8]). Let l be a line in $P = \mathbb{P}\text{Ext}^1(V_2, V_1)$ passing through the point p_0 , where p_0 is the point in P given by (2). If it happens that \mathcal{E}_p is stable for each $p \in l$, then

$$\{\mathcal{E}_p\}_{p \in l}$$

will define a rational curve of degree $2(r, d)$ (with respect to $-K_M$) passing through $[W] \in \mathcal{SU}_C(r, \mathcal{L})$ ([9], [4]). Such a rational curve in $\mathcal{SU}_C(r, \mathcal{L})$ will be called a **rational curve of split type**.

It is known that an extension $0 \rightarrow E \rightarrow W \rightarrow F \rightarrow 0$, where E, W, F are vector bundles on C , gives rise to an element $\delta(W) \in H^1(C, \text{Hom}(F, E))$ which is

the image of the identity homomorphism in $H^0(C, \text{Hom}(F, F))$ by the connecting homomorphism $H^0(C, \text{Hom}(F, F)) \rightarrow H^1(C, \text{Hom}(F, E))$. This gives a one-one correspondence between the set of equivalent classes of extensions of F by E and $H^1(C, \text{Hom}(F, E))$ ([8]).

Lemma 2.1. *Let d be an even number, and $0 \rightarrow L_1 \rightarrow W \rightarrow L_2 \rightarrow 0$ be any non-trivial extension of L_2 by L_1 , where L_1 (resp. L_2) is a line bundle of degree $\frac{d}{2} - 1$ (resp. $\frac{d}{2} + 1$). Then*

(i) *W is semi-stable;*

(ii) *W is non-stable if and only if the element $\delta(W) \in H^1(C, L_2^{-1} \otimes L_1)$ corresponding to W is in the kernel of the map*

$$H^1(C, L_2^{-1} \otimes L_1) \longrightarrow H^1(C, L_2^{-1} \otimes L_1 \otimes L_x),$$

for some $x \in C$. In this case , W is S -equivalent to $L_2 \otimes L_x^{-1} \oplus L_1 \otimes L_x$.

Proof. (i) See Lemma 2.2 in [4] and [5].

(ii) Let L' be a line bundle of degree $\frac{d}{2}$. Then, since $H^0(C, \text{Hom}(L', L_1)) = 0$, it is easy to see that $H^0(C, \text{Hom}(L', W)) \neq 0$ if and only if L' is of the form $L_2 \otimes L_x^{-1}$ for some $x \in C$ and the natural map $L_2 \otimes L_x^{-1} \rightarrow L_2$ can be lifted into a map $L_2 \otimes L_x^{-1} \rightarrow W$.

Consider the commutative diagram of vector bundles

$$\begin{array}{ccccccc} 0 \rightarrow \text{Hom}(L_2, L_1) & \longrightarrow & \text{Hom}(L_2, W) & \longrightarrow & \text{Hom}(L_2, L_2) \rightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow \text{Hom}(L_2 \otimes L_x^{-1}, L_1) & \longrightarrow & \text{Hom}(L_2 \otimes L_x^{-1}, W) & \longrightarrow & \text{Hom}(L_2 \otimes L_x^{-1}, L_2) \rightarrow 0, \end{array}$$

where the horizontal sequences are exact and the vertical maps are induced by the natural map $L_2 \otimes L_x^{-1} \rightarrow L_2$. From this we deduce the commutative diagram

$$\begin{array}{ccccccc} 0 \rightarrow H^0(C, \text{Hom}(L_2, W)) & \longrightarrow & H^0(C, \text{Hom}(L_2, L_2)) & \longrightarrow & H^1(C, \text{Hom}(L_2, L_1)) \rightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow \\ 0 \rightarrow H^0(C, \text{Hom}(L_2 \otimes L_x^{-1}, W)) & \longrightarrow & H^0(C, \text{Hom}(L_2 \otimes L_x^{-1}, L_2)) & \longrightarrow & H^1(C, \text{Hom}(L_2 \otimes L_x^{-1}, L_1)) \rightarrow \cdots \end{array}$$

Which implies the lemma. \square

Remark 2.2. *Lemma 2.1 (ii) asserts that the non-stable bundles in $PH^1(L_2^{-1} \otimes L_1)$ corresponds precisely to the image of C in $PH^1(L_2^{-1} \otimes L_1)$ under the map given by the linear system $K \otimes L_1^{-1} \otimes L_2$. Which implies that the dimension of the subset of non-stable bundles in $PH^1(L_2^{-1} \otimes L_1)$ is at most 1.*

3 The Proof of Theorem 1.3

Let C be a smooth irreducible curve over an algebraically closed field of characteristic zero, W a vector bundle of rank 2 over C , set

$$m(W) := \max\{\deg(L) | L \subset W \text{ is a sub line bundle of } W\}, \quad (3)$$

where the maximum is taken over all sub line bundles L of W . A sub line bundle L of W of maximal degree $m(W)$ is called a **maximal sub line bundle**.

The **Segre invariant** is defined by

$$s(W) := \deg(W) - 2m(W). \quad (4)$$

Note that $s(W) \equiv \deg(W) \pmod{2}$ and that W is stable (resp. semi-stable) if and only if $s(W) \geq 1$ (resp. $s(W) \geq 0$). Nagata proved in [6] that

$$s(W) \leq g.$$

Lemma 3.1. *If $g = 3$, then, for any stable bundle W over C of rank 2 and with even degree d , we have $s(W) = 2$.*

Proof. Since W is stable, we have $1 \leq s(W)$ and $s(W) \leq g = 3$ by Nagata's Theorem ([6]). At the same time, since d is even, it is easy to see that $s(W) \equiv 0 \pmod{2}$ by the definition. Thus we must have $s(W) = 2$. \square

In general, the function $s : M \rightarrow \mathbb{Z}$ defined by $[W] \mapsto s(W)$ is lower semicontinuous and this gives a stratification of M into locally closed subsets M_s according to the value of s . Then, by Proposition 3.1 in [2], we have

Proposition 3.2. ([2]) *Suppose $1 \leq s \leq g - 2$ and $s \equiv d \pmod{2}$. Then M_s is an irreducible algebraic variety of dimension $2g + s - 2$.*

The proof of Theorem 1.3 follows the following two propositions.

Proposition 3.3. *Suppose $g \geq 3$, $r = 2$, d is even and M_2 is non-empty. Then, for any $[W] \in M_2$, there is a rational curve of split type passing through it, which has degree 4.*

Proof. For any $[W] \in M_2$, there is a sub line bundle L_1 of W with $\deg L_1 = \frac{d}{2} - 1$, where $d = \deg \mathcal{L}$. Let $L_2 := W/L_1$ be the quotient bundle, which has degree $\frac{d}{2} + 1$. It is easy to see that

$$1 \times d - \left(\frac{d}{2} - 1\right) \times 2 = 2 = (2, d).$$

Let $i : L_1 \rightarrow W$ be the natural injection, then

$$0 \longrightarrow L_1 \xrightarrow{i} W \longrightarrow L_2 \longrightarrow 0$$

is a non-trivial extension (otherwise, we have $W \cong L_1 \oplus L_2$, which contradicts to the stability of W).

It is known that there is a family of vector bundles \mathcal{E} on C parametrized by $P_{(L_1, L_2)} = \mathbb{P}Ext^1(L_2, L_1)$ so that for each $p \in P_{(L_1, L_2)}$, the \mathcal{E}_p is isomorphic to the bundle obtained as the extension of L_2 by L_1 given by p (see Lemma 2.3 of [8]). More precisely, there is a universal extension

$$0 \rightarrow f^* L_1 \otimes \pi^* \mathcal{O}_{P_{(L_1, L_2)}}(1) \rightarrow \mathcal{E} \rightarrow f^* L_2 \rightarrow 0 \quad (5)$$

on $C \times P_{(L_1, L_2)}$, where $f : C \times P_{(L_1, L_2)} \rightarrow C$ and $\pi : C \times P_{(L_1, L_2)} \rightarrow P_{(L_1, L_2)}$ are projections. Then \mathcal{E} is a family of semi-stable bundles of rank r and with fixed determinant $\det(L_1) \otimes \det(L_2) \cong \mathcal{L}$ (Lemma 2.1). Thus, the universal extension (5) defines a morphism

$$\Phi_{(L_1, L_2)} : P_{(L_1, L_2)} \longrightarrow U_C(2, \mathcal{L}), \quad (6)$$

where $U_C(2, \mathcal{L})$ denotes the moduli space of semi-stable bundles of rank 2 and with fixed determinant \mathcal{L} , which is a projective closure of M .

It is easy to see that $P_{(L_1, L_2)}$ is a projective space of dimension $g \geq 3$. By Lemma 2.1 and Remark 2.2, there is a line l in $P_{(L_1, L_2)}$ passing through

$$q = [0 \longrightarrow L_1 \xrightarrow{i} W \longrightarrow L_2 \longrightarrow 0]$$

such that \mathcal{E}_p is stable for each $p \in l$. Thus, $\Phi_{(L_1, L_2)}(l) \subset M = SU_C(2, \mathcal{L})$ and

$$\Phi_{(L_1, L_2)}|_l : l \rightarrow M = SU_C(2, \mathcal{L}) \quad (7)$$

is a rational curve of split type passing through the point $[W] \in M$. \square

Proposition 3.4. *Suppose $g \geq 2$, $r = 2$, d is odd and M_1 is non-empty. Then, for any $[W] \in M_1$, there is a rational curve of split type passing through it, which has degree 2.*

Proof. Let $[W]$ be a point in M_1 , then we have $s(W) = 1$ and there is a sub line bundle L_1 of W with $\deg L_1 = \frac{d-1}{2}$, where $d = \deg \mathcal{L}$. Let $L_2 := W/L_1$, which is a line bundle of degree $\frac{d+1}{2}$. It is easy to see that

$$1 \times d - \frac{d-1}{2} \times 2 = 1 = (2, d).$$

Let $\iota : L_1 \rightarrow W$ be the natural injection, then

$$0 \longrightarrow L_1 \xrightarrow{\iota} W \longrightarrow L_2 \longrightarrow 0$$

is a non-trivial extension because W is a stable bundle.

It is known that there's a family of vector bundles \mathcal{E} on C parametrized by $P_{(L_1, L_2)} = \mathbb{P}Ext^1(L_2, L_1)$ so that for each $p \in P_{(L_1, L_2)}$, the \mathcal{E}_p is isomorphic to the bundle obtained as the extension of L_2 by L_1 given by p (see Lemma 2.3 of [8]). By Lemma 3.1 of [9], \mathcal{E} is a family of stable bundles of rank 2 and with fixed determinant $\det(L_1) \otimes \det(L_2) \cong \mathcal{L}$, which defines a morphism

$$\Psi_{(L_1, L_2)} : P_{(L_1, L_2)} \longrightarrow SU_C(2, \mathcal{L}) = M. \quad (8)$$

Let l be a line in $P_{(L_1, L_2)}$ passing through

$$q = [0 \longrightarrow L_1 \xrightarrow{\iota} W \longrightarrow L_2 \longrightarrow 0],$$

then

$$\Psi_{(L_1, L_2)}|_l : l \longrightarrow M = SU_C(2, \mathcal{L}) \quad (9)$$

is a rational curve of split type passing through the point $[W] \in M$, which has degree 2. \square

When $g = 2$, the same as Lemma 3.1, we have:

Lemma 3.5. *If $g = 2$, $r = 2$ and d is odd, for any $[W] \in M$, $s(W) = 1$.*

By Lemma 3.5 and Proposition 3.4, we have:

Proposition 3.6. *If $g = 2$, $r = 2$ and d is odd, then, for any $[W] \in M$, there exist a rational curve of split type passing through it, which has degree 2.*

Acknowledgments

The author is grateful to her supervisor Prof. Xiaotao Sun for his helpful suggestions in the preparation of this paper and to Prof. Meng Chen and Prof. Kejian Xu for their help.

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